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It is shown, that the oscillators on a sphere and pseudosphere are related, by the so-called Bohlin transformation, with the Coulomb systems on the pseudosphere: the even states of an oscillator yields the conventional Coulomb system on pseudosphere, while the odd states yield the Coulomb system on pseudosphere in the presence of magnetic flux tube, generating half spin. A similar relation is established for the oscillator on (pseudo)sphere specified by the presence of constant uniform magnetic field B_0 and the Coulomb-like system on pseudosphere specified by the presence of magnetic field $\frac{B}{2r_0}(|\frac{x_3}{\mathbf{x}}| - \epsilon)$. The coorrespondence between oscillator and Coulomb systems in the higher dimensions is also discussed.

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The ($d-$dimensional) oscillator and Coulomb systems are most known representatives of mechanical systems with hidden symmetries which define the $su(d)$ symmetry algebra for the oscillator, and $so(d+1)$ for the Coulomb system. The hidden symmetry has a very transparent meaning in the case of oscillator while in the case of the Coulomb system it has a more complicated interpretation in terms of geodesic flows of a $(d+1)$-dimensional sphere [1]. On the other hand, the transformation $r = R^2$ converts the $(p+1)-$dimensional radial Coulomb problem in $2p-$dimensional radial oscillator one both in classical and quantum cases (the r and R denote the radial coordinates of Coulomb and oscillator systems, respectively). In three distinguished cases, $p = 1, 2, 4$, one can establish the complete correspondence between the Coulomb and the oscillator systems, by using the so-called Bohlin (or Levi-Civita) [2], Kustaanheimo-Stiefel [3] and Hurwitz [4] transformations. This transformations assume the reduction of the oscillator system by the action of Z_2, $U(1)$, $SU(2)$ groups, respectively, and yield the Coulomb-like systems specified by the presence of monopoles [5–7]. The relation of these transformations with the $S^1/Z_2 \cong S^1$, $S^3/S^1 \cong S^2$ and $S^7/S^3 \cong S^4$ bundles (and linearizability of S^1, S^3, S^7 spheres) causes their applications out of the initial prescription and include twistor description of monopoles, instantons and relativistic spinning particles.

On the other hand, the oscillator and Coulomb systems admit the generalizations to a $d-$dimensional sphere and two-sheet hyperboloid (pseudosphere) with radius R_0 given by the potentials [8,9]

$$V_{osc} = \frac{\alpha^2 R_0^2}{2}\frac{\mathbf{x}^2}{x_{d+1}^2}, \quad V_C = -\frac{\gamma}{R_0}\frac{x_{d+1}}{|\mathbf{x}|}, \tag{1}$$

where $\mathbf{x}, x_{d+1}$ are the (pseudo)Euclidean coordinates of ambient space $\mathrm{I\!R}^{d+1}(\mathrm{I\!R}^{d.1})$: $\epsilon\mathbf{x}^2 + x_{d+1}^2 = R_0^2$, $\epsilon = \pm 1$. The $\epsilon = +1$ corresponds to the sphere, $\epsilon = -1$ corresponds to the pseudosphere. These systems possess nonlinear hidden symmetries providing them with the properties similar to those of conventional oscillator and Coulomb systems. Notice that the oscillators on sphere and pseudosphere have the isomorphic configuration spaces (the $d-$dimensional plane with a cut circle, in the stereographic projection), since the first one is undefined on the equator $x_{d+1} = 0$. The Coulomb system is attractive/repulsive on the upper/lower hemisphere, and has the same behavior on both the sheets of the hyperboloid. These systems have been investigated by various methods from many vewpoints (see, e. g. [10] and refs therein).

How to relate the oscillator and Coulomb systems on the (pseudo)spheres ?

This question seems to be crucial for understanding the geometrical meaning of hidden symmetries of Coulomb systems on (pseudo)spheres and for the construction of their generalizations, as well as for the twistor description of the relativistic spinning particles on the AdS spaces. In Ref. [11] devoted to this problem, the oscillator and Coulomb systems on spheres were related by the mappings containing the transitions to imaginary coordinates.

In the present letter, we establish the transparent correspondence between oscillator and Coulomb systems on (pseudo)spheres for the simplest, two-dimensional, case ($p = 1$) that can be extended easily to the higher dimensions ($p = 2, 4$). We show that, in the stereographic projection, the conventional Bohlin transformation relates the two-dimensional oscillator on the (pseudo)sphere with the Coulomb system on pseudosphere, as well as those interacting with specific external magnetic fields. This simple construction allows immediately connect the generators of the hidden symmetry of the systems under consideration, as well as to clarify the mappings suggested in [11].

Let us introduce the complex coordinate z parametrizing the sphere by the complex projective plane $\mathbb{C}\mathrm{P}^1$ and the two-sheeted hyperboloid by the Poincaré disks $\mathcal{L}$:

$$\mathbf{x} \equiv x_1 + ix_2 = R_0\frac{2z}{1+\epsilon z\bar{z}}, \quad x_3 = R_0\frac{1-\epsilon z\bar{z}}{1+\epsilon z\bar{z}}. \tag{2}$$

In these terms the metric takes the Kähler form

$$ds^2 = R_0^2 \frac{4dzd\bar{z}}{(1+\epsilon z\bar{z})^2}, \tag{3}$$

while $R_0 x_k$ define the isometries of the Kahler structure ($su(2)$ if $\epsilon = 1$ and $su(1.1)$ if $\epsilon = -1$). The lower hemisphere and the lower sheet of the hyperboloid are parametrized by the unit disk $|z| < 1$, while the upper hemisphere and the upper sheet of hyperboloid, by its outside, and transform into each other by the inversion $z \to 1/z$. Since in the $R_0 \to \infty$ limit the lower hemisphere (the lower sheet of hyperboloid) converts into the whole two-dimensional plane, for the correspondence with conventional oscillator and Coulomb problems, we have to restrict ourselves by those defined on the lower hemisphere and the lower sheet of hyperboloid (pseudosphere). Notice that the parametrization (2) is nothind else but the stereographic projection of the sphere (if $\epsilon = 1$), and of the two-sheeted hyperboloid (if $\epsilon = -1$):

$$\frac{z}{2R_0} = \begin{cases} \cot\frac{\theta}{2}\mathrm{e}^{i\varphi} & \text{for sphere;} \\ \coth\frac{\theta}{2}\mathrm{e}^{i\varphi} & \text{for pseudosphere,} \end{cases} \tag{4}$$

where θ, φ are the (pseudo)spherical coordinates.

Let us equip the oscillator's phase space $T^*\mathbb{C}\mathrm{P}^1$ ($T^*\mathcal{L}$) by the symplectic structure

$$\omega = d\pi \wedge dz + d\bar{\pi} \wedge d\bar{z} \tag{5}$$

and the rotation generators (defining $su(2)$ algebra if $\epsilon = 1$ and $su(1.1)$ algebra if $\epsilon = -1$)

$$\mathbf{J} \equiv \frac{iJ_1 - J_2}{2} = \pi + \epsilon\bar{z}^2\bar{\pi}, \quad J \equiv \frac{\epsilon J_3}{2} = i(z\pi - \bar{z}\bar{\pi}). \tag{6}$$

These generators, together with $\mathbf{x}/R_0, x_3/R_0$ define the algebra of motion of the (pseudo)sphere via the following nonvanishing Poisson brackets

$$\begin{gathered}\{\mathbf{J}, \mathbf{x}\} = 2x_3, \ \{\mathbf{J}, x_3\} = -\epsilon\bar{\mathbf{x}}, \ \{J, \mathbf{x}\} = i\mathbf{x}, \\ \{\mathbf{J}, \bar{\mathbf{J}}\} = -2i\epsilon J, \ \{\mathbf{J}, J\} = i\mathbf{J}.\end{gathered} \tag{7}$$

In these terms, the Hamiltonian of free particle on (pseudo)sphere reads

$$H_0^\epsilon = \frac{\mathbf{J}\bar{\mathbf{J}} + \epsilon J^2}{2R_0^2} = \frac{(1+\epsilon z\bar{z})^2\pi\bar{\pi}}{2R_0^2} \tag{8}$$

while the oscillator's Hamiltonian is given by the expression

$$H_{osc}^\epsilon(\alpha, R_0|\pi, \bar{\pi}, z, \bar{z}) = \frac{(1+\epsilon z\bar{z})^2\pi\bar{\pi}}{2R_0^2} + \frac{2\alpha^2 R_0^2 z\bar{z}}{(1-\epsilon z\bar{z})^2}. \tag{9}$$

The latter system possesses the hidden symmetry given by the complex (or vectorial) constant of motion [9]

$$\mathbf{I} = I_1 + iI_2 = \frac{\mathbf{J}^2}{2R_0^2} + \frac{\alpha^2 R_0^2}{2}\frac{\bar{\mathbf{x}}^2}{x_3^2}, \tag{10}$$

which defines, together with J and H_{osc} , the cubic algebra

$$\{\mathbf{I}, J\} = 2i\mathbf{I}, \ \ \{\bar{\mathbf{I}}, \mathbf{I}\} = 4i\left(\alpha^2 J + \frac{\epsilon J H_{osc}}{R_0^2} - \frac{J^3}{2R_0^4}\right). \tag{11}$$

The energy surface of the oscillator on the (pseudo)sphere $H_{osc}^\epsilon = E$ reads

$$\frac{\left(1-(z\bar{z})^2\right)^2\pi\bar{\pi}}{2R_0^4} + 2\left(\alpha^2 + \epsilon\frac{E}{R_0^2}\right)z\bar{z} = \frac{E}{R_0^2}\left(1+(z\bar{z})^2\right). \tag{12}$$

Now, performing the canonical Bohlin transformation [2]

$$w = z^2, \quad p = \frac{\pi}{2z}, \tag{13}$$

we convert the energy surface of the oscillator (12) onto the one of the Coulomb system on the pseudosphere:

$$\frac{(1-w\bar{w})^2 p\bar{p}}{2r_0^2} - \frac{\gamma}{r_0}\frac{1+w\bar{w}}{2|w|} = \mathcal{E}_C, \tag{14}$$

where

$$r_0 = R_0^2, \quad \gamma = \frac{E}{2}, \quad -2\mathcal{E}_C = \alpha^2 + \epsilon\frac{E}{r_0}. \tag{15}$$

The constants of motion of the oscillators, J and $\mathbf{I}$ (which are equal on the energy surfaces (12)) converted, respectively into the doubled angular momentum and the doubled Runge-Lenz vector of the Coulomb system

$$J \to 2J_C, \quad \mathbf{I} \to 2\mathbf{A}, \quad \mathbf{A} = -\frac{iJ_C\mathbf{J}_C}{r_0} + \gamma\frac{\bar{\mathbf{x}}_C}{|\mathbf{x}_C|}, \tag{16}$$

where $\mathbf{J}_C$, J_C, $\mathbf{x}_C$ denote the rotation generators and the pseudo-Euclidean coordinates of the Coulomb system. *Hence, the Bohlin transformation of the classical isotropic oscillator on the (pseudo)sphere yields the classical Coulomb problem on the pseudosphere.*

The quantum-mechanical counterpart of the energy surface (12) is the Schrödinger equation

$$\mathcal{H}_{osc}^\epsilon(\alpha, R_0|\pi, \bar{\pi}, z, \bar{z})\Psi(z, \bar{z}) = E\Psi(z, \bar{z}), \tag{17}$$

with the quantum Hamiltonian defined (due to the two-dimensional origin of the system) by the expression (9), where $\pi, \bar{\pi}$ are the momenta operators (hereafter we assume $\hbar = 1$)

$$\pi = -i\frac{\partial}{\partial z}, \quad \bar{\pi} = -i\frac{\partial}{\partial\bar{z}}. \tag{18}$$

The energy spectrum of this system is given by the expression (see e.g. [10] and refs therein)

$$E = \tilde{\alpha}(N+1) + \epsilon\frac{(N+1)^2}{2R_0^2}, \quad N = 2n_r + |M|, \tag{19}$$

where $\tilde\alpha=\sqrt{\alpha^2+1/(4R_0^4)}$, M is the eigenvalue of J, N is the principal quantum number, n_r is the radial quantum number,

$$|M|, n_r \leq N, \quad M,\ n_r,\ N=1,\ldots,N_{max} \\ N_{max}=\left\{\begin{array}{ll}\infty\ , & \text{if } \epsilon=1\\ [\tilde\alpha R_0^2]-1\ , & \text{if } \epsilon=-1\end{array}\right. \tag{20}$$

So, the the number of levels in the energy spectrum of the oscillator is infinite on the sphere and finite on the pseudosphere. The degeneracy of the energy spectrum is the same as in the flat case, $2N+1$.

The quantum-mechanical correspondence between oscillator and Coulomb systems is more complicated, because the Bohlin transformation (13) maps the z-plane into the two-sheeted Riemann surface, since arg $w\in[0,4\pi)$. Thus, we have to supply the quantum-mechanical Bohlin transformation with the reduction by the Z_2 group action, choosing either even ($\sigma=0$) or odd ($\sigma=1/2$) wave functions

$$\Psi_\sigma(z,\bar z)=\psi_\sigma(z^2,\bar z^2)(z/\bar z)^{2\sigma}: \\ \psi_\sigma(|w|,\arg w+2\pi)=\psi_\sigma(|w|,\arg w). \tag{21}$$

This implies that the range of definition of w can be restricted, without loss of generality, to arg $w\in[0,2\pi)$. In that case, the resulting system is the Coulomb problem on the hyperboloid given by the Schrödinger equation

$$H^-_C(\gamma,r|p_\sigma,\bar p_\sigma,w,\bar w)\psi_\sigma=\mathcal{E}_C\psi_\sigma \tag{22}$$

where $\gamma,\mathcal{E}_C,r$ are given by (15), and the momenta operators are of the form

$$p_\sigma=-i\frac{\partial}{\partial w}-\frac{\sigma}{2iw},\quad \bar p_\sigma=-i\frac{\partial}{\partial\bar w}+\frac{\sigma}{2i\bar w}. \tag{23}$$

Hence, the resulting Coulomb system includes the interaction with the magnetic vortex (an infinitely thin solenoid) with the magnetic flux $\pi\sigma$ and zero strength $rot\sigma/w=0$. Such a composites are tipical representatives of the anyonic systems with the spin σ [12]. *So, we get a conventional 2d Coulomb problem on the hyperboloid at $\sigma=0$ and those with half spin generated by the magnetic flux, at $\sigma=1/2$.* Taking into account the relations (15), one can rewrite the oscillator's energy spectrum (19) as follows

$$\sqrt{\frac{1}{4r_0^2}-\epsilon\frac{2\gamma}{r_0}-2\mathcal{E}_C}=\frac{2\gamma}{N+1}-\epsilon\frac{N+1}{2r_0}. \tag{24}$$

From this expression one can easily obtain the energy spectrum of the reduced system on the pseudosphere

$$\mathcal{E}_C=-\frac{N_\sigma(N_\sigma+1)}{2r_0^2}-\frac{\gamma^2}{2(N_\sigma+1/2)^2}, \tag{25}$$

where

$$N_\sigma=N/2=n_r+m_\sigma,\quad m_\sigma=m/2, \\ n_r, m_\sigma-\sigma, N_\sigma-\sigma=0,1,\ldots,N_\sigma^{max}-\sigma. \tag{26}$$

Here m_σ denotes the eigenvalue of the angular momentum of the reduced system, and n_r is the radial quantum number of the initial (and reduced) system. Notice, that the magnetic vortex shifts the energy levels of the two-dimensional Coulomb system which is nothing else than the reflection of Aharonov-Bohm effect.
It is seen, that the whole spectrum of the oscillator on pseudosphere ($\epsilon=-1$) transforms in the spectra of the constructed Coulomb systems on the pseudosphere, while for the oscillator on the sphere ($\epsilon=1$) the positivity of l. h. s. of (24) restrict the admissible values of N_σ. So, only the part of the spectrum of the oscillator on the sphere transforms into the spectrum of Coulomb system. Hence, in both cases we get the same result

$$0\leq N_\sigma^{max}\leq\sqrt{r_0\gamma}-1/2. \tag{27}$$

To obtain the flat limit we make the rescaling

$$(z,\ \pi)\to(\frac{z}{2R_0},\ 2R_0\pi),\quad (w,\ p)\to(\frac{w}{4r_0},\ 4r_0p),$$

where $r_0=R_0^2$, and then take the limit $R_0\to\infty$. In this limit, the oscillator on the (pseudo)sphere results in the conventional circular oscillator

$$H=2\pi\bar\pi+\frac{\alpha^2z\bar z}{2},\quad \omega=d\pi\wedge dz+d\bar\pi\wedge d\bar z, \tag{28}$$

which possesses the hidden $su(2)$ symmetry given by the constants of motion

$$J=i(\pi z-\bar\pi\bar z),\quad \mathbf{I}=2\pi^2+\alpha^2\bar z^2/2: \\ \{\bar{\mathbf{I}},\mathbf{I}\}=4i\alpha^2J,\quad \{\mathbf{I},J\}=2i\mathbf{I}. \tag{29}$$

The canonical transformation (13) remains unchanged, the energy level of oscillator converts into the energy level of the Coulomb problem with coupling constant $E\ /2$ and the energy $-\alpha^2/2$. The oscillator's constants of motion J and $\mathbf{I}$ yield, respectively, the doubled angular momentum and the doubled Runge-Lenz vector

$$\mathbf{A}=-4ipJ+\frac{E}{2}\frac{\bar w}{|w|}.$$

In the quantum case, the even states of the oscillator yield the conventional Coulomb system, while the odd states of the oscillator yield the Coulomb systemin the precence of magnetic vortex generating the half spin [5].

Let us briefly discuss the Bohlin transformation for the oscillator on the (pseudo)sphere interacting with constant magnetic field B_0. This system can be defined by the following replacement of the symplectic structure (5) and of the rotation generators (6)

$$\omega\to\omega+B_0\frac{i4R_0^2dz\wedge d\bar z}{(1+\epsilon z\bar z)^2},\quad J_i\to J_i+4R_0B_0x_i, \tag{30}$$

which shifts the initial Hamiltonian (9) on $(4B_0)^2$.
Since the replacement (30) preserves the algebra (7),

the modified system also possesses the hidden symmetry given by (10), (11). The Bohlin transformation (13) of the modified oscillator yields the Coulomb system on the pseudosphere interacting with the magnetic field

$$B_C = \frac{B}{2r_0}\left(\frac{x_{(C)3}}{|\mathbf{x}|} - \epsilon\right). \tag{31}$$

It is easy to see, that the $2p-$dimensional oscillator on (pseudo)sphere can be connected with the $(p+1)-$dimensional Coulomb-like systems on pseudosphere in the same manner in the higher dimensions ($p = 2, 4$). Indeed, in stereographic coordinates, the oscillator on $2p$-dimensional (pseudo)sphere is described by the Hamiltonian system given by (5), (9), where the following replacement is performed $(z,\ \pi) \to (z^a,\ \pi_a)$, $a = 1, \ldots, p$ with the summation over these indices. Consequently, the oscillator's energy surfaces are of the form (12). Further reduction to the $(p+1)$-dimensional Coulomb-like system on pseudosphere repeats the corresponding reduction in the flat case [6,7].

For example, if $p = 1$, we reduce the system under consideration by the Hamiltonian action of $U(1)$ group given by the generator

$$J = i(z\pi - \bar{z}\bar{\pi}).$$

For this purpose, we have to fix the level surface

$$J = 2s$$

and choose the $U(1)$-invariant stereographic coordinates in the form of conventional Kustaanheimo-Stiefel transformation [3] (see also [6])

$$\mathbf{y} = z\sigma\bar{z}, \quad \mathbf{p} = \frac{z\sigma\pi + \bar{\pi}\sigma\bar{z}}{2(z\bar{z})}, \tag{32}$$

where σ are Pauli matrices.
As a result, the reduced symplectic structure reads

$$d\mathbf{y} \wedge d\mathbf{p} + s\frac{\mathbf{y} \times d\mathbf{y} \times \wedge d\mathbf{y}}{|\mathbf{y}|^3}, \tag{33}$$

while the oscillator's energy surface takes the form

$$\frac{(1-y^2)^2}{2r_0^2}(p^2 + \frac{s^2}{y^2}) - \frac{\gamma}{r_0}\frac{1-y^2}{2y} = \mathcal{E}_C, \tag{34}$$

where $y = |\mathbf{y}|$, $\mathbf{y}$ denote the stereographic coordinates of three-dimentional pseudosphere, and

$$r_0 = R_0^2, \quad \gamma = 2E, \quad -\mathcal{E}_C = 2(\alpha^2 + \epsilon\frac{E}{r_0}), \tag{35}$$

So, we get the energy surface of the pseudospherical analog of a Coulomb-like system proposed in Ref. [13] describing the interaction of two non-relativistic dyons.

In $p = 4$ case, we have to reduce the system by the action of $SU(2)$ group

$$z^a \to z^a g, \ \ g\bar{g} = 1, \quad g \in \mathbb{H}, \ z^a \in \mathbb{H}^2$$

and choose the $SU(2)-$invariant stereographic coordinates and momenta in the form corresponding to the standard Hurwitz transformation [4,7]

$$w = 2z_1\bar{z}_2, \quad x_5 = z_1\bar{z}_1 - z_2\bar{z}_2, \tag{36}$$

which yields a pseudospherical analog of the five-dimensional Coulomb-like system with $SU(2)$ monopole [7].

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